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Connection between complete and Möbius forms of gauge invariant operators *

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Abstract

We study the connection between complete representations of gauge invariant operators and their Möbius representations acting in a limited space of functions. The possibility to restore the complete representations from Möbius forms in the coordinate space is proven and a method of restoration is worked out. The operators for transition from the standard BFKL kernel to the quasi-conformal one are found both in Möbius and total representations.

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1 Introduction

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) approach [1] was formulated in momentum space. In this space the kernel of the BFKL equation was calculated in the next-to-leading order (NLO) long ago, at first for forward scattering (i.e. for $t = 0$ and color singlet in the t-channel) [2] and then for any fixed (not growing with energy) momentum transfer t and any possible two-gluon color state in the t-channel [3]. Unfortunately, the NLO kernel is rather complicated. In the colour singlet case at $t \neq 0$ it consists of numerous intricate two-dimensional integrals.

In the case, most interesting for phenomenological applications, of colourless particles scattering, the leading order (LO) BFKL kernel has a remarkable property [4]. In the Möbius representation, i.e. in the space of functions vanishing at coinciding transverse coordinates (impact parameters) of Reggeons, it is invariant with regard to conformal transformations of these coordinates. Moreover, in this representation the kernel coincides [5] with the kernel of the colour dipole model [6] formulated in the impact parameter space.

In the NLO one could expect that in the Möbius representation the BFKL kernel would be quasi-conformal, i.e. its conformal invariance would be violated only by terms proportional to β-function, so that it would remain unbroken in N=4 supersymmetric Yang-Mills theory (N=4 SUSY). However, the direct transformation of the known QCD kernel from the momentum into the impact parameter space, with subsequent transition into the Möbius representation, gives a kernel which is not quasi-conformal [5, 7, 8]. In N=4 SUSY the conformal invariance of the kernel obtained in such a way is also broken [9]. Then, the kernel of the colour dipole model was calculated in the NLO directly in the impact parameter space [10, 11]. It turned out that the result differs from the one obtained by transformation from the momentum space.

However there is an ambiguity in the definition of low-x evolution kernels at NLO. It is analogous to the well-known ambiguity of NLO anomalous dimensions and follows from the possibility to redistribute radiative corrections between kernels and impact factors. This ambiguity was discussed in detail in Ref. [12]. It has been shown recently [13] that it permits one both to reach agreement with the colour dipole model (with account of the improvement of the result in Ref. [11] made in Ref. [14]) and to obtain quasi-conformal shape for the kernel in the Möbius representation. This shape appears to be

quite simple. It is unbelievably simple in comparison with the known shape of the kernel in the momentum space. We will call "standard" the kernel and the impact factors defined in the NLO in Ref. [15] in the space of transverse momenta of two interacting Reggeons. Evidently, the question arises about the relation between these two shapes.

Transition from the "complete" to the Möbius representation means restriction of the complete space of functions where the kernel is defined to the space of functions which are equal to zero at coinciding values of Reggeon impact parameters. Therefore, the interrelation between the latter representation and the complete one is not obvious. In particular, the possibility to restore the complete representation from the Möbius one is questionable. Our paper is devoted to the discussion of this problem. It is organized as follows. In the next Section all necessary notations and definitions are given. In Section 3 the equivalence of complete and Möbius representations for gauge invariant operators is proven. In Section 4 the complete representation of the operator connecting standard and quasi-conformal kernels is restored from its Möbius representation. In Section 5 the interrelation of complete and Möbius representations is illustrated on the example of this operator. Conclusions are drawn in Section 6.

2 Notation and definitions

For brevity we will use, as in Ref. [5], states $|\vec{q}\rangle$ with definite two-dimensional (we will not use dimensional regularization and put the space-time dimension $D = 4$) transverse Reggeon momentum $\vec{q}$ and states $|\vec{r}\rangle$ with definite Reggeon impact parameter $\vec{r}$ normalized as follows:

$$\langle\vec{q}|\vec{q}\,'\rangle = \delta(\vec{q} - \vec{q}\,')\ , \qquad \langle\vec{r}|\vec{r}\,'\rangle = \delta(\vec{r} - \vec{r}\,')\ , \qquad \langle\vec{r}|\vec{q}\rangle = \frac{e^{i\vec{q}\vec{r}}}{2\pi}\ . \tag{1}$$

The kernel defined in Ref. [15] is represented by an operator $\hat{K}$. It is given by the sum of virtual (related to the gluon Regge trajectory) and real (related to real particle production in Reggeon collisions) parts, so that

$$\hat{K} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{K}_r\ . \tag{2}$$

Here 1 and 2 are Reggeon indices,

$$\langle\vec{q}_i|\hat{\omega}_i|\vec{q}_i\,'\rangle = \delta(\vec{q}_i - \vec{q}_i\,')\omega(-\vec{q}_i^{\,2})\ , \tag{3}$$

$\omega(t)$ is called the gluon trajectory (although, in fact, the trajectory is $1+\omega(t)$), and

$$\langle\vec{q}_1,\vec{q}_2|\hat{K}_r|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle = \delta(\vec{q}_1+\vec{q}_2-\vec{q}_1^{\,\prime}-\vec{q}_2^{\,\prime})\frac{1}{\sqrt{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}}K_r(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q})\frac{1}{\sqrt{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,\prime 2}}}\;, \quad (4)$$

where $\vec{q}=\vec{q}_1+\vec{q}_2$ and $K_r(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q})$ is defined in Ref. [15]. The appearance of square roots can seem strange to an experienced reader; these square roots are connected with the normalization (1) and can be removed by passing to states $|\vec{q}\rangle$ normalized as $\langle\vec{q}|\vec{q}^{\,\prime}\rangle=\vec{q}^{\,2}\delta(\vec{q}-\vec{q}^{\,\prime})$. In the LO

$$K_r^{(B)}(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q}) = \frac{\alpha_s N_c}{2\pi^2}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}}-\vec{q}^{\,2}\right)\;, \quad (5)$$

where $\vec{k}=\vec{q}_1-\vec{q}_1^{\,\prime}=\vec{q}_2^{\,\prime}-\vec{q}_2$ and the superscript (B) denotes leading order.

In terms of the kernel $\hat{K}$, the s-channel discontinuities of scattering amplitudes for processes $A+B\to A'+B'$ are presented as

$$-4i(2\pi)^2\delta(\vec{q}_A-\vec{q}_B)\mathrm{disc}_s\mathcal{A}_{AB}^{A'B'} = \langle A'\bar{A}|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}e^{Y\hat{K}}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}|\bar{B}'B\rangle\;, \quad (6)$$

where $Y=\ln(s/s_0)$, s_0 is an appropriate energy scale, $q_A=p_{A'}-p_A$, $q_B=p_B-p_{B'}$. The states $\langle A'\bar{A}|$ and $|\bar{B}'B\rangle$ are normalized in such a way that

$$\langle\vec{q}_1,\vec{q}_2|\bar{B}'B\rangle = 4p_B^-\delta(\vec{q}_B-\vec{q}_1-\vec{q}_2)\Phi_{B'B}(\vec{q}_1,\vec{q}_2)\;, \quad (7)$$

$$\langle A'\bar{A}|\vec{q}_1,\vec{q}_2\rangle = 4p_A^+\delta(\vec{q}_A-\vec{q}_1-\vec{q}_2)\Phi_{A'A}(\vec{q}_1,\vec{q}_2)\;, \quad (8)$$

with $p^\pm=(p_0\pm p_z)/\sqrt{2}$ and the impact factors Φ expressed through the Reggeon vertices according to Ref. [15].

The kernel $\hat{K}$ is symmetric, as it can be seen from Eqs. (2)–(5), i.e. $\hat{K}=\hat{K}^T$ or

$$\langle\vec{q}_1,\vec{q}_2|\hat{K}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle = \langle\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}|\hat{K}|\vec{q}_1,\vec{q}_2\rangle\;. \quad (9)$$

However, the kernel which is conformally invariant in the Möbius representation in the LO [4, 16] is not $\hat{K}$, but the non-symmetric kernel

$$\hat{\mathcal{K}} = \left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}\;\hat{K}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{\frac{1}{2}}. \quad (10)$$

The transition to this kernel is possible thanks to the invariance of the discontinuity (6) with respect to the transformation

$$\hat{K}\to\hat{\mathcal{O}}^{-1}\hat{K}\hat{\mathcal{O}}\;,\qquad \langle A'\bar{A}|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}\to\langle A'\bar{A}|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}\hat{\mathcal{O}}\;,$$

$$\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}|\bar{B}'B\rangle \to \hat{\mathcal{O}}^{-1}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}|\bar{B}'B\rangle\ , \tag{11}$$

with any non-singular operator $\hat{\mathcal{O}}$. Taking $\hat{\mathcal{O}} = \left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{1/2}$ we get (10) and the right-hand side of the discontinuity (6) becomes

$$\langle A'\bar{A}|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}} e^{Y\hat{K}}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-\frac{1}{2}}|\bar{B}'B\rangle\ = \langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle\ . \tag{12}$$

It is important that, after setting the kernel $\mathcal{K}$ by Eq. (10), which provides conformal invariance of its Möbius representation in the LO, an additional transformation with $\hat{\mathcal{O}} = 1 - \alpha_s\hat{U}$ is still possible. With NLO accuracy it gives

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)},\hat{U}]\ ,\ \ \langle A'\bar{A}| \to \langle A'\bar{A}| - \left(\langle A'\bar{A}|\right)^{(B)}\alpha_s\hat{U},$$

$$\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle \to \left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle + \alpha_s\hat{U}\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}\left(|\bar{B}'B\rangle\right)^{(B)}. \tag{13}$$

It was shown [13] that the transformation (13) permits one to remove the discrepancy between the BFKL and the colour dipole kernels (with account of the correction of the result of Ref. [11] made in Ref. [14]), and to obtain the kernel

$$\hat{\mathcal{K}}^{QC} = \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)},\hat{U}] \tag{14}$$

which is quasi-conformal in the Möbius representation.

The operator $\hat{U}$ was found as the sum of two pieces, $\hat{U} = \hat{U}_1 + \hat{U}_2$. The first piece was found in the momentum space,

$$\langle\vec{q}_1,\vec{q}_2|\alpha_s\hat{U}_1|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle = \frac{\alpha_s N_c}{2\pi^2}\Bigg[-\delta(\vec{q}-\vec{q}^{\,\prime})\left(\frac{\vec{k}}{\vec{k}^{\,2}}-\frac{\vec{q}_1}{\vec{q}_1^{\,2}}\right)\left(\frac{\vec{k}}{\vec{k}^{\,2}}+\frac{\vec{q}_2}{\vec{q}_2^{\,2}}\right)\ln\vec{k}^{\,2}$$

$$+\delta(\vec{q}_1-\vec{q}_1^{\,\prime})\delta\left(\vec{q}_2-\vec{q}_2^{\,\prime}\right)\left(\int d^2l\left(\frac{1}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{2\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{2\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^2}\right)\ln\vec{l}^{\,2}\right.$$

$$\left.\left.-\frac{\pi\beta_0}{4N_c}\ln\left(\vec{q}_1^{\,2}\vec{q}_2^{\,2}\right)\right)\right], \tag{15}$$

where $\vec{q} = \vec{q}_1+\vec{q}_2$, $\vec{q}^{\,\prime} = \vec{q}_1^{\,\prime}+\vec{q}_2^{\,\prime}$, β_0 is the first coefficient of the β-function, and $\vec{k} = \vec{q}_1-\vec{q}_1^{\,\prime}$. Note that the integral over $\vec{l}$ diverges in $\vec{l}=0$ and, strictly

speaking, the term with $1/\vec{l}^{\,2}$ must be regularized. But in fact what we need is the action of the operator U_1 on some state, i.e. the integral over $\vec{k}$ of the product of the matrix element and a function of $\vec{k}$, rather than the matrix element itself. In this integral the singularities at $\vec{l} = 0$ and $\vec{k} = 0$ cancel and we get a finite result (evidently, the terms with $1/\vec{l}^{\,2}$ and $1/\vec{k}^{\,2}$ must be regularized in the same way). The second part

$$\langle \vec{r}_1 \vec{r}_2 | \alpha_s \hat{U}_{2M} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s N_c}{4\pi^2} \int d\vec{r}_0 \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2} \vec{r}_{02}^{\,2}} \ln \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2} \vec{r}_{02}^{\,2}} \right)$$

$$\times \left[\delta(\vec{r}_{11'}) \delta(\vec{r}_{02'}) + \delta(\vec{r}_{01'}) \delta(\vec{r}_{22'}) - \delta(\vec{r}_{11'}) \delta(r_{22'}) \right] , \qquad (16)$$

was found in the impact parameter space and in the Möbius representation, which is indicated by the subscript M (hereafter $\vec{r}_{ij'} = \vec{r}_i - \vec{r}_j{}'$).

Thus, the quasi-conformal kernel $\hat{\mathcal{K}}^{QC}$ determined by Eqs. (14)–(16) was found in the impact parameter space and in the Möbius representation. Now its explicit form is known in this space and this representation for theories containing fermions and scalars in arbitrary representations of the colour group [17].

Remind that transition to the Möbius representation means truncation of the space of states. Therefore, the connection between operators in this representation and in the complete space of states (i.e. in the "complete" representation) is not obvious. In particular, it is not clear if it is possible to restore $\hat{U}_2$ (and consequently $\hat{\mathcal{K}}^{QC}$) in the complete representation in the momentum space from Eq. (16). In the next Section we prove the possibility of such restoration.

3 Interrelation between complete and Möbius representations

The possibility to rebuild the complete kernel from its Möbius representation is based on the gauge invariance of the kernel. Note that this property, together with the gauge invariance of impact factors for colourless particles, was used for the transition to the Möbius representation in Ref. [4]. Only thanks to this property the discontinuity (6) can be written using the Möbius

representation of $\hat{\mathcal{K}}$. Let us remind how the passage to the Möbius representation was done.

Gauge invariance of impact factors means that

$$\langle A'\bar{A}|\vec{q},0\rangle = \langle A'\bar{A}|0,\vec{q}\rangle = \langle \vec{q},0|\bar{B}'B\rangle = \langle 0,\vec{q}|\bar{B}'B\rangle = 0\ , \tag{17}$$

while gauge invariance of the kernel $\hat{K}$ implies the property

$$K_r(\vec{q}_1,\vec{q}_1{}';\vec{q})|_{\vec{q}_1=0} = K_r(\vec{q}_1,\vec{q}_1{}';\vec{q})|_{\vec{q}_1{}'=0}$$

$$= K_r(\vec{q}_1,\vec{q}_1{}';\vec{q})|_{\vec{q}_1=\vec{q}} = K_r(\vec{q}_1,\vec{q}_1{}';\vec{q})|_{\vec{q}_1{}'=\vec{q}} = 0\ . \tag{18}$$

As we may easily see from Eqs. (2), (4), and (6), just these properties guarantee the absence of Coulomb divergences in the discontinuities.

From Eqs. (10), (4) and these properties, it also follows that

$$\langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}|\vec{q},0\rangle = \langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}|0,\vec{q}\rangle = 0\ . \tag{19}$$

It means that $\langle A'\bar{A}|e^{Y\hat{\mathcal{K}}}|\Psi\rangle = 0$ if $\langle \vec{r}_1,\vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$. Then, Eq. (12) shows that one can make the replacement

$$\langle \vec{r}_1,\vec{r}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle \to \langle \vec{r}_1,\vec{r}_2|\left(\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle\right)_M = \langle \vec{r}_1,\vec{r}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle$$

$$-\frac{1}{2}\langle \vec{r}_1,\vec{r}_1|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle - \frac{1}{2}\langle \vec{r}_2,\vec{r}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle\ , \tag{20}$$

retaining the discontinuity (6). Evidently, this substitution transfers the state $\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle$ into the Möbius representation. Note that with the requirement of symmetry of the state with respect to the Reggeon exchange this transformation is unique. In the momentum space it reads

$$\langle \vec{q}_1,\vec{q}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle_M = \langle \vec{q}_1,\vec{q}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle$$

$$-\frac{1}{2}\left(\delta(\vec{q}_1-\vec{q}_B)\delta(\vec{q}_2)+\delta(\vec{q}_2-\vec{q}_B)\delta(\vec{q}_1)\right)\int d\vec{l}_1 d\vec{l}_2 \langle \vec{l}_1,\vec{l}_2|\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)^{-1}|\bar{B}'B\rangle\ . \tag{21}$$

Then one can transfer from $\mathcal{K}$ to $\mathcal{K}_M$ without changing the discontinuity. This is done omitting in $\mathcal{K}$ both the terms which are zero in the Möbius subspace and the terms whose action on any state in the Möbius subspace puts it out of this subspace. Note that the last procedure is not unique.

Indeed, the kernel $\langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}',\vec{r}_2{}'\rangle$ in the impact parameter space can be written as

$$\langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}',\vec{r}_2{}'\rangle = \langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}|\vec{r}_1{}',\vec{r}_2{}'\rangle_t - f_1(\vec{r}_{11'},\vec{r}_{12'}) - f_2(\vec{r}_{21'},\vec{r}_{22'}) \ , \quad (22)$$

where the subscript t means omitting the terms proportional to $\delta(\vec{r}_{1'2'})$, and functions f_1 and f_2 are restricted (besides the absence of the terms proportional to $\delta(\vec{r}_{1'2'})$) by the requirement

$$f_1(\vec{r}_{01'},\vec{r}_{02'}) + f_2(\vec{r}_{01'},\vec{r}_{02'}) = \langle\vec{r}_0,\vec{r}_0|\hat{\mathcal{K}}|\vec{r}_1{}',\vec{r}_2{}'\rangle_t \,. \quad (23)$$

But the uncertainty in the choice of f_1 and f_2 plays no role due to the symmetry of the impact factors $\langle A'\bar{A}|$ with respect to the Reggeon exchange. Indeed, if we have two sets of functions $f_i^{(1)}$ and $f_i^{(2)}$, $i=1,2$, satisfying (23), then the difference

$$\left[f_1^{(1)}(\vec{r}_{11'},\vec{r}_{12'}) + f_2^{(1)}(\vec{r}_{21'},\vec{r}_{22'})\right] - \left[f_1^{(2)}(\vec{r}_{11'},\vec{r}_{12'}) + f_2^{(2)}(\vec{r}_{21'},\vec{r}_{22'})\right]$$

is antisymmetric with respect to replacement $\vec{r}_1 \leftrightarrow \vec{r}_2$. In fact, this uncertainty can be used for the simplification of $\langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}',\vec{r}_2{}'\rangle$. On the other hand, if one does not like the uncertainty, it can be removed imposing the requirement of the corresponding symmetry on the kernel, so that in the following we will not pay attention to it.

Thus, in the impact parameter space the Möbius representation of the kernel $\mathcal{K}$ is unambiguously constructed from the complete one, assuming the symmetry with respect to the Reggeon exchange (which follows from the boson nature of Reggeons). Evidently, this statement is valid for any operator defined both in the momentum and the coordinate spaces.

Note that, strictly speaking, operators in the Möbius representation are not defined in the momentum space. The reason is that in the impact parameter space they can be singular at $\vec{r}_{1'2'}=0$, so that their direct transformation into the momentum space can be impossible. Using translation invariance, we can formally write

$$\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}_M|\vec{q}_1{}',\vec{q}_2{}'\rangle = \int\frac{d\vec{r}_1}{2\pi}\frac{d\vec{r}_2}{2\pi}\frac{d\vec{r}_1{}'}{2\pi}\frac{d\vec{r}_2{}'}{2\pi}e^{-i\vec{q}_1\vec{r}_1-i\vec{q}_2\vec{r}_2+i\vec{q}_1{}'\vec{r}_1{}'+i\vec{q}_2{}'\vec{r}_2{}'}\langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1{}',\vec{r}_2{}'\rangle$$

$$= \delta(\vec{q}_1+\vec{q}_2-\vec{q}_1{}'-\vec{q}_2{}')\mathcal{K}_M(\vec{q}_1,\vec{q}_2;\vec{k}) \ , \quad (24)$$

where $\vec{k} = \vec{q}_1 - \vec{q}_1^{\,\prime} = \vec{q}_2^{\,\prime} - \vec{q}_2$ and

$$\mathcal{K}_M(\vec{q}_1, \vec{q}_2; \vec{k}) = \int \frac{d\vec{r}_{11'}}{2\pi}\frac{d\vec{r}_{22'}}{2\pi} d\vec{r}_{1'2'} e^{-i\vec{q}_1\vec{r}_{11'} - i\vec{q}_2\vec{r}_{22'} - i\vec{k}\vec{r}_{1'2'}} \langle \vec{r}_1, \vec{r}_2 | \hat{\mathcal{K}}_M | \vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime} \rangle \ , \tag{25}$$

(not to be confused with $K_r(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q})$). But the integral over $\vec{r}_{1'2'}$ in (25) can be divergent because of singularities of the type $\ln^n \vec{r}^{\,2}_{1'2'}/\vec{r}^{\,2}_{1'2'}$. It is useful to understand that singularities at $\vec{r}_{1'2'} = 0$ are related with the growth of operators in momentum space at large $\vec{k}^{\,2}$. Thus, the NLO BFKL kernel in momentum space contains terms with $\ln \vec{k}^{\,2}$ and $\ln^2 \vec{k}^{\,2}$ behaviour at large $\vec{k}^{\,2}$. At fixed $\vec{r}_{1'2'} \neq 0$ we have

$$\int \frac{d\vec{k}}{2\pi} e^{i\vec{k}\,\vec{r}_{1'2'}} \ln(\vec{k}^{\,2}) = -\frac{2}{\vec{r}^{\,2}_{1'2'}} \ ,$$

$$\int \frac{d\vec{k}}{2\pi} e^{i\vec{k}\,\vec{r}_{1'2'}} \ln^2(\vec{k}^{\,2}) = \frac{4}{\vec{r}^{\,2}_{1'2'}} \left(\ln\left(\frac{\vec{r}^{\,2}_{1'2'}}{4}\right) - 2\psi(1) \right) \quad . \tag{26}$$

This result can be obtained, for example, writing

$$\ln^n \vec{k}^{\,2} = (-1)^n \frac{d^n}{d\alpha^n} \, (\vec{k}^{\,2})^{-\alpha} \Big|_{\alpha=0}$$

and using the equality

$$\int \frac{d\vec{k}}{2\pi} (\vec{k}^{\,2})^{-\alpha} e^{i\vec{k}\,\vec{r}} = \frac{2}{\vec{r}^{\,2}} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \alpha \left(\frac{\vec{r}^{\,2}}{4}\right)^{\alpha} . \tag{27}$$

In fact, the limits $\alpha \to 0$ and $\vec{r}^{\,2} \to 0$ are not interchangeable. It means that result (26) cannot be used at arbitrary small $\vec{r}^{\,2}_{1'2'}$. In the Möbius representation small $\vec{r}^{\,2}_{1'2'}$ are unimportant and the result (26) is used everywhere. But in the integral (25) the singularity at $\vec{r}^{\,2}_{1'2'}$ must be regularized. From the consideration above it is natural to use in (25), instead of $(1/\vec{r}^{\,2})$ and $(\ln \vec{r}^{\,2}/\vec{r}^{\,2})$, the regularized functions $(1/\vec{r}^{\,2})_R$ and $(\ln \vec{r}^{\,2}/\vec{r}^{\,2})_R$ which make possible the inverse Fourier transform,

$$\int \frac{d\vec{r}}{2\pi} \left(\frac{1}{\vec{r}^{\,2}}\right)_R e^{-i\vec{k}\,\vec{r}} = -\frac{1}{2} \ln(\vec{k}^{\,2}) \ ,$$

$$\int \frac{d\vec{r}}{2\pi} \left(\frac{1}{\vec{r}^{\,2}} \left[\ln\left(\frac{\vec{r}^{\,2}}{4}\right) - 2\psi(1) \right] \right)_R e^{-i\vec{k}\,\vec{r}} = \frac{1}{4} \ln^2(\vec{k}^{\,2}) \ . \tag{28}$$

Since

$$\int \frac{d\vec{r}}{2\pi}\frac{e^{-i\vec{k}\,\vec{r}}}{\vec{r}^{\,2}}\theta(\vec{r}^{\,2}-c^2)|_{c\to 0} = -\frac{1}{2}\left(\ln\left(\frac{\vec{k}^{\,2}}{4}\right) - 2\psi(1) + \ln c^2\right) ,$$

$$\int \frac{d\vec{r}}{2\pi}\frac{e^{-i\vec{k}\,\vec{r}}}{\vec{r}^{\,2}}\ln\vec{r}^{\,2}\theta(\vec{r}^{\,2}-c^2)|_{c\to 0} = \frac{1}{4}\left(\left(\ln\left(\frac{\vec{k}^{\,2}}{4}\right) - 2\psi(1)\right)^2 - \ln^2 c^2\right) , \tag{29}$$

this can be done defining $1/(\vec{r}^{\,2})_R$ and $(\ln\vec{r}^{\,2}/\vec{r}^{\,2})$ at $\vec{r}^{\,2}\to 0$ in the following way:

$$\int \frac{d\vec{r}}{2\pi}\left(\frac{1}{\vec{r}^{\,2}}\right)_R \theta(c^2-\vec{r}^{\,2})|_{c\to 0} = \frac{1}{2}\left(\ln c^2 - 2\psi(1) - \ln 4\right) ,$$

$$\int \frac{d\vec{r}}{2\pi}\left(\frac{\ln\vec{r}^{\,2}}{\vec{r}^{\,2}}\right)_R \theta(c^2-\vec{r}^{\,2})|_{c\to 0} = \frac{1}{4}\left(\ln^2 c^2 - (2\psi(1)+\ln 4)^2\right) . \tag{30}$$

For definiteness, let us accept that (25) is defined with such regularization. But, in fact, the choice of a regularization is not important for the restoration of the complete kernel from the Möbius one. Indeed, since any regularization concerns only the region $\vec{r}_{1'2'}\to 0$, any change of regularization has an influence only on terms not depending on $\vec{k}$. Therefore, denoting

$$\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle = \delta(\vec{q}_1+\vec{q}_2-\vec{q}_1^{\,\prime}-\vec{q}_2^{\,\prime})\mathcal{K}(\vec{q}_1,\vec{q}_2;\vec{k}) , \tag{31}$$

we obtain from Eq. (22)

$$\mathcal{K}(\vec{q}_1,\vec{q}_2;\vec{k})_- = \mathcal{K}_M(\vec{q}_1,\vec{q}_2;\vec{k})_- + \delta(\vec{q}_2)\int d\vec{r}_{11'}d\vec{r}_{12'}e^{-i(\vec{q}_1-\vec{k})\vec{r}_{11'}-i\vec{k}\vec{r}_{12'}}f_1(\vec{r}_{11'},\vec{r}_{12'})$$

$$+\,\delta(\vec{q}_1)\int d\vec{r}_{21'}d\vec{r}_{22'}e^{-i(\vec{q}_2+\vec{k})\vec{r}_{22'}+i\vec{k}\vec{r}_{21'}}f_2(\vec{r}_{21'},\vec{r}_{22'}) , \tag{32}$$

where the subscript $\mathcal{K}(\vec{q}_1,\vec{q}_2;\vec{k})_-$ means $\mathcal{K}(\vec{q}_1,\vec{q}_2;\vec{k})$ without terms independent of the third argument in $\mathcal{K}(\vec{q}_1,\vec{q}_2;\vec{k})$.

As was stated before, the Möbius representation of the BFKL kernel is unambiguously constructed from the complete one, by requiring the symmetry with respect to the Reggeon exchange. But the inverse statement is also valid. The Möbius representation of the BFKL kernel totally defines the complete kernel symmetric with respect to Reggeon exchange. For the

validity of this statement two properties of the complete BFKL kernel are important. The first one is its gauge invariance which gives (see Eqs. (10), (18) and (31))

$$\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{q}_1) = \mathcal{K}_r(\vec{q}_1, \vec{q}_2; -\vec{q}_2) = 0 \ . \tag{33}$$

And the second one is the absence of terms proportional to $\delta(\vec{q}_1)$ or $\delta(\vec{q}_2)$ in the kernel. This fixes the residual freedom connected with such terms.

These properties provide (with account of the symmetrization discussed above) the uniqueness of the restoration of the total kernel from its Möbius representation. Indeed, if there were two different complete kernels $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ with the same Möbius representation, then the Möbius representation for their difference would be zero. It follows from Eq. (32) that in this case it must be

$$\mathcal{K}^{(1)}(\vec{q}_1, \vec{q}_2; \vec{k})_- - \mathcal{K}^{(2)}(\vec{q}_1, \vec{q}_2; \vec{k})_- = 0, \tag{34}$$

i.e. they can differ only in terms independent of $\vec{k}$. On the other hand, gauge invariance requires turning the difference $\mathcal{K}^{(1)}(\vec{q}_1, \vec{q}_2; \vec{k})_- - \mathcal{K}^{(2)}(\vec{q}_1, \vec{q}_2; \vec{k})_-$ into zero at $\vec{k} = \vec{q}_1$ and at $\vec{k} = -\vec{q}_2$. Therefore, it is zero identically.

Thus, the uniqueness of the restoration of $\hat{\mathcal{K}}$ from $\hat{\mathcal{K}}_M$ is proven. This proof and (32) give the way to perform the restoration.

4 Restoration of the operator U_2 from its Möbius form

Let us demonstrate the restoration of the complete operator from its Möbius form on the example of the operator U_2 given in Eq. (16). First of all, it is necessary to note that $\hat{U}$ in the transformation (13) cannot be arbitrary if we want to conserve the possibility to use the Möbius representation after this transformation. Indeed, in this case the transformation (13) must conserve the gauge invariance of the impact factor $\langle A'\bar{A}|$ and kernel $\hat{\mathcal{K}}$. Therefore, $\hat{U}$ must be gauge invariant in the same way as $\hat{\mathcal{K}}$. Moreover, without any loss of generality we can consider that it has no terms proportional to $\delta(\vec{q}_1)$ or $\delta(\vec{q}_2)$ in the momentum space, since such terms do not contribute to the discontinuity (6). In other words, $\hat{U}$ has the same properties as $\hat{\mathcal{K}}$. For the part $\hat{U}_1$ these properties are easily seen from Eq. (15). It means that $\hat{U}_2$ has the same properties and therefore can be unambiguously restored from Eq. (16).

To do that, let us first transfer Eq. (16) into the momentum space. As was pointed above, generally an operator $\hat{\mathcal{O}}$ in the Möbius representation in the impact parameter space can contain non-integrable singularities at $\vec{r}_{1'2'} = 0$ which require regularization. But, as we may notice from Eq. (16), $\langle\vec{r}_1\vec{r}_2|\hat{U}_{2M}|\vec{r}_1{}'\vec{r}_2{}'\rangle$ does not have explicit singularities at $\vec{r}_{1'2'} = 0$, though its separate parts are divergent. Therefore we will treat the sum of these parts and calculate directly $\langle\vec{q}_1\vec{q}_2|\hat{U}_{2M}|\vec{q}_1{}'\vec{q}_2{}'\rangle$.

Here we meet technical problems related to the separation of real and virtual parts, as usually occurs when one operates with the BFKL kernel. Again looking at Eq. (16), we observe that there are terms in $\langle\vec{r}_1\vec{r}_2|\hat{U}_{2M}|\vec{r}_1{}'\vec{r}_2{}'\rangle$ which have ultraviolet singularities at $\vec{r}_{01} = 0$ and $\vec{r}_{02} = 0$ that cancel in their sum. We will treat them together. But this is not the only problem. Another problem is that, quite analogously to $\mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{k})$, $U_2(\vec{q}_1, \vec{q}_2; \vec{k})$ (defined by Eq. (31) with the substitution $\mathcal{K} \to U_2$) contains singularities at $\vec{k} = 0$ in the virtual (proportional to $\delta(\vec{k})$) and the real parts. They cancel each other; but to make this cancellation evident one needs to write the coefficient of $\delta(\vec{k})$ in an integral form.

Defining $U_{2M}(\vec{q}_1, \vec{q}_2; \vec{k})$ according to Eqs. (24) and (25) with the substitution $\hat{\mathcal{K}}_M \to \hat{U}_{2M}$, using Eq. (16), and integrating the delta-functions, we obtain

$$\alpha_s U_{2M}(\vec{q}_1, \vec{q}_2; \vec{k}) = \frac{\alpha_s N_c}{4\pi^2} \int \frac{d\vec{r}_1}{2\pi} \frac{d\vec{r}_2}{2\pi} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right)$$
$$\times \left[e^{-i\vec{k}\,\vec{r}_1 - i\vec{q}_2\,\vec{r}_2} + e^{-i\vec{q}_1\,\vec{r}_1 + i\vec{k}\,\vec{r}_2} - e^{-i\vec{k}\,\vec{r}_{12}}\right] . \tag{35}$$

Let us divide $\hat{U}_2$ into two pieces, $\hat{U}_2 = \hat{U}_2^r + \hat{U}_2^v$; making in Eq. (35) the following decomposition:

$$\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right) = \frac{1}{\vec{r}_1^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_2^{\,2}}\right) + \frac{1}{\vec{r}_2^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}}\right) - 2\frac{\vec{r}_1\vec{r}_2}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right)$$
$$+ \frac{1}{\vec{r}_1^{\,2}} \ln\left(\frac{1}{\vec{r}_1^{\,2}}\right) + \frac{1}{\vec{r}_2^{\,2}} \ln\left(\frac{1}{\vec{r}_2^{\,2}}\right) , \tag{36}$$

the first three terms in the decomposition correspond to $\hat{U}_2^r$ and the last two ones correspond to $\hat{U}_2^v$. Then $U_{2M}^r(\vec{q}_1, \vec{q}_2; \vec{k})$ is calculated using the integrals

$$\int \frac{d\vec{r}_1}{2\pi} \frac{d\vec{r}_2}{2\pi} \frac{1}{\vec{r}_1^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_2^{\,2}}\right) e^{-i\vec{a}\,\vec{r}_1 - i\vec{b}\,\vec{r}_2} = \frac{1}{\vec{b}^{\,2}} \ln\left(\frac{(\vec{a}+\vec{b})^2}{\vec{a}^{\,2}}\right) ,$$

$$\int \frac{d\vec{r}_1}{2\pi}\frac{d\vec{r}_2}{2\pi}\frac{\vec{r}_1\vec{r}_2}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_1^{\,2}}\right)e^{-i\vec{a}\,\vec{r}_1-i\vec{b}\,\vec{r}_2} = -\frac{\vec{a}\vec{b}}{\vec{a}^{\,2}\vec{b}^{\,2}}\ln\left(\frac{(\vec{a}+\vec{b})^2}{\vec{b}^{\,2}}\right),$$

$$\int \frac{d\vec{r}}{2\pi}e^{-i\vec{a}\,\vec{r}}\frac{\vec{r}}{\vec{r}^{\,2}} = \frac{-i\vec{a}}{\vec{a}^{\,2}}\,, \tag{37}$$

$$\int \frac{d\vec{r}}{2\pi}e^{-i\vec{a}\,\vec{r}}\frac{\vec{r}}{\vec{r}^{\,2}}\ln(\vec{r}^{\,2}) = \frac{-i\vec{a}}{\vec{a}^{\,2}}\left(2\psi(1)-\ln\left(\frac{\vec{a}^{\,2}}{4}\right)\right)\,,$$

with the result

$$\alpha_s U_{2M}^r(\vec{q}_1,\vec{q}_2,\vec{k}) = \frac{\alpha_s N_c}{4\pi^2}\left[\frac{2}{\vec{k}^{\,2}}\ln(\vec{k}^{\,2})+\frac{1}{\vec{q}_1^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}}{\vec{k}^{\,2}}\right)\right.$$

$$+\frac{1}{\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}_2^{\,\prime 2}}{\vec{k}^{\,2}}\right)+\frac{1}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,\prime 2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)-2\frac{\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}}\ln\left(\vec{q}_1^{\,\prime 2}\right)+2\frac{\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\ln\left(\vec{q}_2^{\,\prime 2}\right)$$

$$\left.-2\left(\psi(1)+\ln 2\right)\left(\frac{2}{\vec{k}^{\,2}}-\frac{2\vec{q}_1\vec{k}}{\vec{q}_1^{\,2}\vec{k}^{\,2}}+\frac{2\vec{q}_2\vec{k}}{\vec{q}_2^{\,2}\vec{k}^{\,2}}\right)\right]\,, \tag{38}$$

where $\vec{q}_1^{\,\prime} = \vec{q}_1-\vec{k},\ \vec{q}_2^{\,\prime} = \vec{q}_2+\vec{k}$.

The last two terms into the decomposition (36) after integration (35) give contributions proportional to $\delta(\vec{q}_1)$, $\delta(\vec{q}_2)$ and $\delta(k)$ with divergent coefficients. The terms proportional to $\delta(\vec{q}_1)$ and $\delta(\vec{q}_2)$ can be omitted. The coefficient of $\delta(k)$ is presented in an integral form using the trick [5]

$$\int d\vec{r}\,\frac{1}{\vec{r}^{\,2}}\ln\left(\frac{1}{\vec{r}^{\,2}}\right)\left(e^{i\vec{q}_1\,\vec{r}}+e^{i\vec{q}_2\,\vec{r}}-2\right)$$

$$= \int d\vec{l}\,\frac{d\vec{r}_1}{2\pi}\,\frac{d\vec{r}_2}{2\pi}\,\frac{\vec{r}_1\vec{r}_2}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}e^{-i\,\vec{l}\,(\vec{r}_1+\vec{r}_2)}\left[e^{i\vec{q}_1\,\vec{r}_1}\ln(\vec{r}_2^{\,2})+e^{i\vec{q}_2\,\vec{r}_2}\ln(\vec{r}_1^{\,2})-\ln(\vec{r}_1^{\,2}\vec{r}_2^{\,2})\right]$$

$$= \int d\vec{l}\left[2\psi(1)+\ln 4-\ln(\vec{l}^{\,2})\right]\left(\frac{2}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^2}\right)\,. \tag{39}$$

At last, the real part (38) must be changed by adding terms independent of $\vec{k}$ in such a way that it becomes zero at $\vec{q}_1^{\,\prime} = 0$ and $\vec{q}_2^{\,\prime} = 0$. The result is

$$\langle\vec{q}_1,\vec{q}_2|\alpha_s\hat{U}_2|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle = \delta(\vec{q}_{11'}+\vec{q}_{22'})\frac{\alpha_s N_c}{4\pi^2}\left[\frac{2}{\vec{k}^{\,2}}\ln(\vec{k}^{\,2})+\frac{1}{\vec{q}_1^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)\right.$$

$$
+\frac{1}{\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}_2^{\,\prime 2}\vec{q}_1^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)+\frac{1}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,\prime 2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)-2\frac{\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}}\ln\left(\vec{q}_1^{\,\prime 2}\right)+2\frac{\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\ln\left(\vec{q}_2^{\,\prime 2}\right)
$$

$$
-2\frac{\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\ln(\vec{q}^{\,2})\Bigg]-\frac{\alpha_s N_c}{4\pi^2}\delta(\vec{q}_{22'})\delta\left(\vec{q}_{11'}\right)\int d\vec{l}\ln\vec{l}^{\,2}\left(\frac{2}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}\right.
$$

$$
\left.-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^2}\right)-(\psi(1)+\ln 2)\,\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}^B|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle\ , \tag{40}
$$

where

$$
\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}^{(B)}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle=\delta(\vec{q}_{11'}+\vec{q}_{22'})\frac{\alpha_s N_c}{2\pi^2}\left[\frac{2}{\vec{k}^{\,2}}-2\frac{\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}}+2\frac{\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\right.
$$

$$
\left.-2\frac{\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}-\delta(\vec{k})\int d\vec{l}\left(\frac{2}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^{\,2}}\right)\right]. \tag{41}
$$

Evidently, the last term in Eq. (40) does not contribute to the commutator $[\hat{\mathcal{K}}^{(B)},\hat{U}_2]$ and therefore can be omitted.

For the full operator $\hat{U}=\hat{U}_1+\hat{U}_2$ we have from Eqs. (15) and (40)

$$
\langle\vec{q}_1,\vec{q}_2|\alpha_s\hat{U}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle=\delta(\vec{q}_{11'}+\vec{q}_{22'})\frac{\alpha_s N_c}{4\pi^2}\left[\frac{1}{\vec{q}_1^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)+\frac{1}{\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}_2^{\,\prime 2}\vec{q}_1^{\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\right)\right.
$$

$$
\left.+\frac{1}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,\prime 2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)-\frac{2\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}}\ln\left(\frac{\vec{q}_1^{\,\prime 2}}{\vec{k}^{\,2}}\right)+\frac{2\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}_2^{\,\prime 2}}{\vec{k}^{\,2}}\right)-\frac{2\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\ln\left(\frac{\vec{q}^{\,2}}{\vec{k}^{\,2}}\right)\right]
$$

$$
-\frac{\alpha_s\beta_0}{8\pi}\ln\left(\vec{q}_1^{\,2}\vec{q}_2^{\,2}\right)\delta(\vec{q}_{11'})\delta(\vec{q}_{22'})-(\psi(1)+\ln 2)\,\langle\vec{q}_1,\vec{q}_2|\hat{K}^{(B)}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle\ . \tag{42}
$$

Note that, except the term with $\hat{\mathcal{K}}^{(B)}$ (which can be omitted), it does not have the virtual part at all.

5 Möbius representation for the operator $\hat{U}$

We have restored the complete operator $\hat{U}_2$ from its Möbius form and have obtained the total operator $\hat{U}$ in the momentum space. But sometimes the Möbius representation can be more convenient than the complete one. Therefore in this Section we will construct the Möbius representation for $\hat{U}$. Since

$\hat{U}_2$ was originally written in this representation (16), we have to find the Möbius form for $\hat{U}_1$. As was already mentioned, $\hat{U}_1$ has the same properties (gauge invariance and absence of terms proportional to $\delta(\vec{q}_1)$ or $\delta(\vec{q}_2)$) as $\hat{\mathcal{K}}$. According to the prescription (22), first we need to find

$$\langle\vec{r}_1,\vec{r}_2|\alpha_s\hat{U}_1|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle = \frac{\alpha_s N_c}{4\pi^2}\int\frac{d\vec{q}_1}{2\pi}\frac{d\vec{q}_2}{2\pi}\frac{d\vec{k}}{(2\pi)^2}e^{i\vec{q}_1\vec{r}_{11'}+i\vec{q}_2\vec{r}_{22'}+i\vec{k}\vec{r}_{1'2'}}\Bigg[-\frac{2}{\vec{k}^{\,2}}\ln\vec{k}^{\,2}$$

$$+2\left(\frac{\vec{k}\vec{q}_1}{\vec{k}^{\,2}\vec{q}_1^{\,2}}-\frac{\vec{k}\vec{q}_2}{\vec{k}^{\,2}\vec{q}_2^{\,2}}+\frac{\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)\ln\vec{k}^{\,2}+\delta(\vec{k})\left(-\frac{\pi\beta_0}{2N_c}\ln\left(\vec{q}_1^{\,2}\vec{q}_2^{\,2}\right)\right.$$

$$\left.\left.+\int d^2l\left(\frac{2}{\vec{l}^{\,2}}-\frac{\vec{l}(\vec{l}-\vec{q}_1)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_1)^2}-\frac{\vec{l}(\vec{l}-\vec{q}_2)}{\vec{l}^{\,2}(\vec{l}-\vec{q}_2)^2}\right)\ln\vec{l}^{\,2}\right)\right]. \tag{43}$$

Since in the integrand (in the square brackets) there are no terms independent of $\vec{k}$, after the integration there will be no terms proportional to $\delta(\vec{r}_{1'2'})$ which should be omitted. In principle, the next steps are defined in the prescription (22). But as in the preceding Section, one must face the technical problems of separation of real and virtual parts, since they are separately singular. In Eq. (43) the first term in the square brackets and the first term in the integral over $\vec{l}$ are separately infrared singular and must be treated together. These and only these terms give a contribution proportional to $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ in the impact parameter space (and can be called "virtual" in this space). But the second of them is not only infrared, but also ultraviolet divergent, so that the coefficient of $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ contains an ultraviolet singularity. Therefore, it must be written in an integral form using the same trick as before:

$$\int\frac{d\vec{q}_1}{2\pi}\frac{d\vec{q}_2}{2\pi}\frac{d\vec{k}}{(2\pi)^2}e^{i\vec{q}_1\vec{r}_{11'}+i\vec{q}_2\vec{r}_{22'}+i\vec{k}\vec{r}_{1'2'}}\left(-\frac{2}{\vec{k}^{\,2}}\ln\vec{k}^{\,2}+\delta(\vec{k})\int d^2l\frac{2}{\vec{l}^{\,2}}\ln\vec{l}^{\,2}\right)$$

$$=-\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\int\frac{d\vec{k}}{\vec{k}^{\,2}}\ln\vec{k}^{\,2}\left(2e^{i\vec{k}\vec{r}_{1'2'}}-2\right)=\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\int d\vec{r}_0\frac{d\vec{k}_1}{2\pi}\frac{d\vec{k}_2}{2\pi}$$

$$\times\frac{\vec{k}_1\vec{k}_2}{\vec{k}_1^{\,2}\vec{k}_2^{\,2}}\left(e^{i\vec{k}_1\vec{r}_{01}+i\vec{k}_2\vec{r}_{02}}\ln(\vec{k}_1^{\,2}\vec{k}_2^{\,2})-e^{i(\vec{k}_1+\vec{k}_2)\vec{r}_{01}}\ln(\vec{k}_1^{\,2})-e^{i(\vec{k}_1+\vec{k}_2)\vec{r}_{02}}\ln(\vec{k}_2^{\,2})\right)$$

$$=\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\int d\vec{r}_0\Bigg[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\left(2\psi(1)+\ln 4\right)$$

$$+\frac{\vec r_{01}\vec r_{02}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\vec r_{01}^{\,2}\vec r_{02}^{\,2}\right)-\frac{1}{\vec r_{01}^{\,2}}\ln\left(\vec r_{01}^{\,2}\right)-\frac{1}{\vec r_{02}^{\,2}}\ln\left(\vec r_{02}^{\,2}\right)\Bigg]\,, \tag{44}$$

where we used the integrals (37). Evidently, the representation (44) is not unique. Using the equality

$$\int d\vec r_0\Big[\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}{(\vec r_{12}^{\,2})^2}\right)-\left(\frac{1}{\vec r_{01}^{\,2}}-\frac{1}{\vec r_{02}^{\,2}}\right)\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\Big]=0\,, \tag{45}$$

we come to the representation

$$\int\frac{d\vec q_1}{2\pi}\frac{d\vec q_2}{2\pi}\frac{d\vec k}{(2\pi)^2}e^{i\vec q_1\vec r_{11'}+i\vec q_2\vec r_{22'}+i\vec k\vec r_{1'2'}}\left(-\frac{2}{\vec k^{\,2}}\ln\vec k^{\,2}+\delta(\vec k)\int d^2l\frac{2}{\vec l^{\,2}}\ln\vec l^{\,2}\right)$$

$$=\delta(\vec r_{11'})\delta(\vec r_{22'})\int d\vec r_0\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\Big[(2\psi(1)+\ln 4)+\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\Big]\,. \tag{46}$$

The ultraviolet divergence in this (virtual in impact parameter space) contribution must cancel analogous divergences of the other terms. Their calculation does not require any trick. Using the integrals (37), we obtain

$$\int\frac{d\vec q_1}{2\pi}\frac{d\vec q_2}{2\pi}\frac{d\vec k}{(2\pi)^2}e^{i\vec q_1\vec r_{11'}+i\vec q_2\vec r_{22'}+i\vec k\vec r_{1'2'}}\left[\frac{2\vec q_1\vec k}{\vec q_1^{\,2}\vec k^{\,2}}\ln\vec k^{\,2}+\delta(\vec k)\int d^2l\frac{(\vec q_1-\vec l)\vec l}{(\vec q_1-\vec l)^2\vec l^{\,2}}\ln\vec l^{\,2}\right]$$

$$=\delta(\vec r_{22'})\left[-\frac{\vec r_{12}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{21'}^{\,2}}\Big(2\psi(1)+\ln 4-\ln\left(\vec r_{21'}^{\,2}\right)\Big)+\frac{1}{\vec r_{11'}^{\,2}}\ln\left(\frac{\vec r_{11'}^{\,2}}{\vec r_{21'}^{\,2}}\right)\right.$$

$$\left.+\frac{1}{\vec r_{21'}^{\,2}}\left(2\psi(1)+\ln 4-\ln(\vec r_{21'}^{\,2})\right)\right]\,. \tag{47}$$

As follows from the representation (22), terms in the last line do not contribute to the Möbius representation and can be omitted. The terms in Eq. (43) corresponding to those in the square brackets at the L.H.S. of Eq. (47) after the substitution $\vec q_1\leftrightarrow\vec q_2,\vec k\leftrightarrow-\vec k$ give a contribution equal to the R.H.S. of Eq. (47) after the substitution $\vec r_1\leftrightarrow\vec r_2,\vec r_1^{\,\prime}\leftrightarrow\vec r_2^{\,\prime}$. To calculate the remaining terms of (43), one needs, besides the integrals (37), only the Fourier transform of $\ln\vec q^{\,2}$. Since it is singular, it requires regularization (i.e.

extension of the definition). It was considered in detail in Section 3. Choosing the functions f_1 and f_2 in Eq. (22) from consideration of simplicity, we obtain

$$\langle \vec r_1 \vec r_2|\alpha_s \hat U_{1M}|\vec r_1{}' \vec r_2{}'\rangle = \frac{\alpha_s N_c}{4\pi^2}\int d\vec r_0 \left\{ \delta(\vec r_{11'})\delta(\vec r_{02'}) \left[\frac{\vec r_{12}^{\,2}\ln(\vec r_{01}^{\,2})}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}} + \frac{1}{\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}}{\vec r_{01}^{\,2}}\right)\right]\right.$$

$$\left. +\delta(\vec r_{22'})\delta(\vec r_{01'}) \left[\frac{\vec r_{12}^{\,2}\ln(\vec r_{02}^{\,2})}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}} + \frac{1}{\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\right] + \delta(\vec r_{11'})\delta(r_{22'})\frac{\vec r_{12}^{\,2}\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right\}$$

$$+\frac{1}{\pi \vec r_{1'2'}^{\,2}}\left[\frac{2\vec r_{11'}\vec r_{22'}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}} - \frac{\vec r_{11'}\vec r_{12'}}{\vec r_{11'}^{\,2}\vec r_{12'}^{\,2}} - \frac{\vec r_{21'}\vec r_{22'}}{\vec r_{21'}^{\,2}\vec r_{22'}^{\,2}}\right] - (\psi(1)+\ln 2)\langle \vec r_1, \vec r_2|\hat K_M^{(B)}|\vec r_1{}', \vec r_2{}'\rangle$$

$$+\frac{\alpha_s\beta_0}{8\pi^2}\left[\delta(\vec r_{11'})\left(\frac{1}{(\vec r_{22'}^{\,2})_R} - \frac{1}{\vec r_{12'}^{\,2}}\right) + \delta(\vec r_{22'})\left(\frac{1}{(\vec r_{11'}^{\,2})_R} - \frac{1}{\vec r_{21'}^{\,2}}\right)\right], \qquad (48)$$

where $(1/\vec r^{\,2})_R$ is defined as in (28), (30) and $\hat K_M^{(B)}$ is the leading order BFKL kernel in the Möbius representation

$$\langle \vec r_1 \vec r_2|\hat{\mathcal K}_M^{(B)}|\vec r_1{}'\vec r_2{}'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int d\vec r_0 \frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}$$

$$\times\left[\delta(\vec r_{11'})\delta(\vec r_{02'}) + \delta(\vec r_{01'})\delta(\vec r_{22'}) - \delta(\vec r_{11'})\delta(r_{22'})\right]. \qquad (49)$$

The Möbius form of the total operator $\hat U$ is given by the sum of the two pieces expressed in Eqs. (48) and (16):

$$\langle \vec r_1 \vec r_2|\alpha_s \hat U_M|\vec r_1{}'\vec r_2{}'\rangle = \frac{\alpha_s N_c}{4\pi^2}\int d\vec r_0\left\{\delta(\vec r_{11'})\delta(\vec r_{02'})\left[\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{02}^{\,2}}\right)\right.\right.$$

$$\left.\left.+\frac{1}{\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}}{\vec r_{01}^{\,2}}\right)\right] + \delta(\vec r_{22'})\delta(\vec r_{01'})\left[\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}}\right) + \frac{1}{\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\right]\right\}$$

$$+\frac{1}{\pi \vec r_{1'2'}^{\,2}}\left[\frac{2\vec r_{11'}\vec r_{22'}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}} - \frac{\vec r_{11'}\vec r_{12'}}{\vec r_{11'}^{\,2}\vec r_{12'}^{\,2}} - \frac{\vec r_{21'}\vec r_{22'}}{\vec r_{21'}^{\,2}\vec r_{22'}^{\,2}}\right] - (\psi(1)+\ln 2)\langle \vec r_1, \vec r_2|\hat K_M^{B}|\vec r_1{}', \vec r_2{}'\rangle$$

$$+\frac{\alpha_s\beta_0}{8\pi^2}\left[\delta(\vec r_{11'})\left(\frac{1}{(\vec r_{22'}^{\,2})_R} - \frac{1}{\vec r_{12'}^{\,2}}\right) + \delta(\vec r_{22'})\left(\frac{1}{(\vec r_{11'}^{\,2})_R} - \frac{1}{\vec r_{21'}^{\,2}}\right)\right]. \qquad (50)$$

It is worthwhile to mention that transferring Eq. (42) into the Möbius representation in the impact parameter space in accordance with the prescription (22) gives exactly the result (50).

6 Conclusion

We investigated the connection between the complete and Möbius representations of gauge invariant operators, taking particular care of the BFKL kernel for scattering of colourless particles. Following Ref. [16] we call Möbius representation (form) of some two-particle (or two-Reggeon, as in the case of BFKL kernel) operator its form in the space of functions vanishing at coinciding impact parameters of these particles. In this representation the BFKL kernel has remarkable properties. In the leading order it is invariant with respect to the group of Möbius transformations of impact parameters [4], and in the NLO it can be transformed into a simple quasi-conformal shape. An important question is the possibility of restoration of the complete kernel from this shape. It is evident that a generic operator cannot be completely restored from its Möbius representation, since in this representation it acts in a truncated space of functions. Moreover, the explicit form of a generic operator in this representation can be written only in the coordinate space. The transformation into the momentum space can be impossible because of the singularity at coinciding impact parameters. However, this is not the case for the BFKL kernel, for which Möbius and complete representations are equivalent. The reason for that is the gauge invariance of the kernel.

In this paper it has been shown that for any gauge invariant two-particle operator it is possible to restore the complete operator from its Möbius representation. We have shown that the restoration is unique up to terms proportional to $\delta(\vec{q}_1)$ or $\delta(\vec{q}_2)$ and symmetry with respect to the Reggeon exchange. It was explicitly demonstrated for the operator responsible for the transformation of the standard BFKL kernel to the quasi-conformal shape. Originally this operator was presented as a sum of two pieces, one of them was found acting in the complete representation in the momentum space and the other in the Möbius representation in the coordinate space. We found both Möbius and complete representations of the full operator.

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